First order phase transitions in $Sp(2N)$

Swansea University Prifysgol Abertawe
Technical University of Munich

Fabian Zierler

with E.Bennett, B.Lucini, D.Mason, M.Piai, E.Rinaldi, D.Vadacchino

within the TELOS collaboration

Multi-canonical Methods and Lattice Field Theory

December 1 2025, ECT* Trento

based on: `2305.07463`, `2409.19426`, `2509.19009`

code, data, analysis: DOI 10.5281/zenodo.13807993 DOI 10.5281/zenodo.16580109 DOI 10.5281/zenodo.16579683

Gravitational Waves

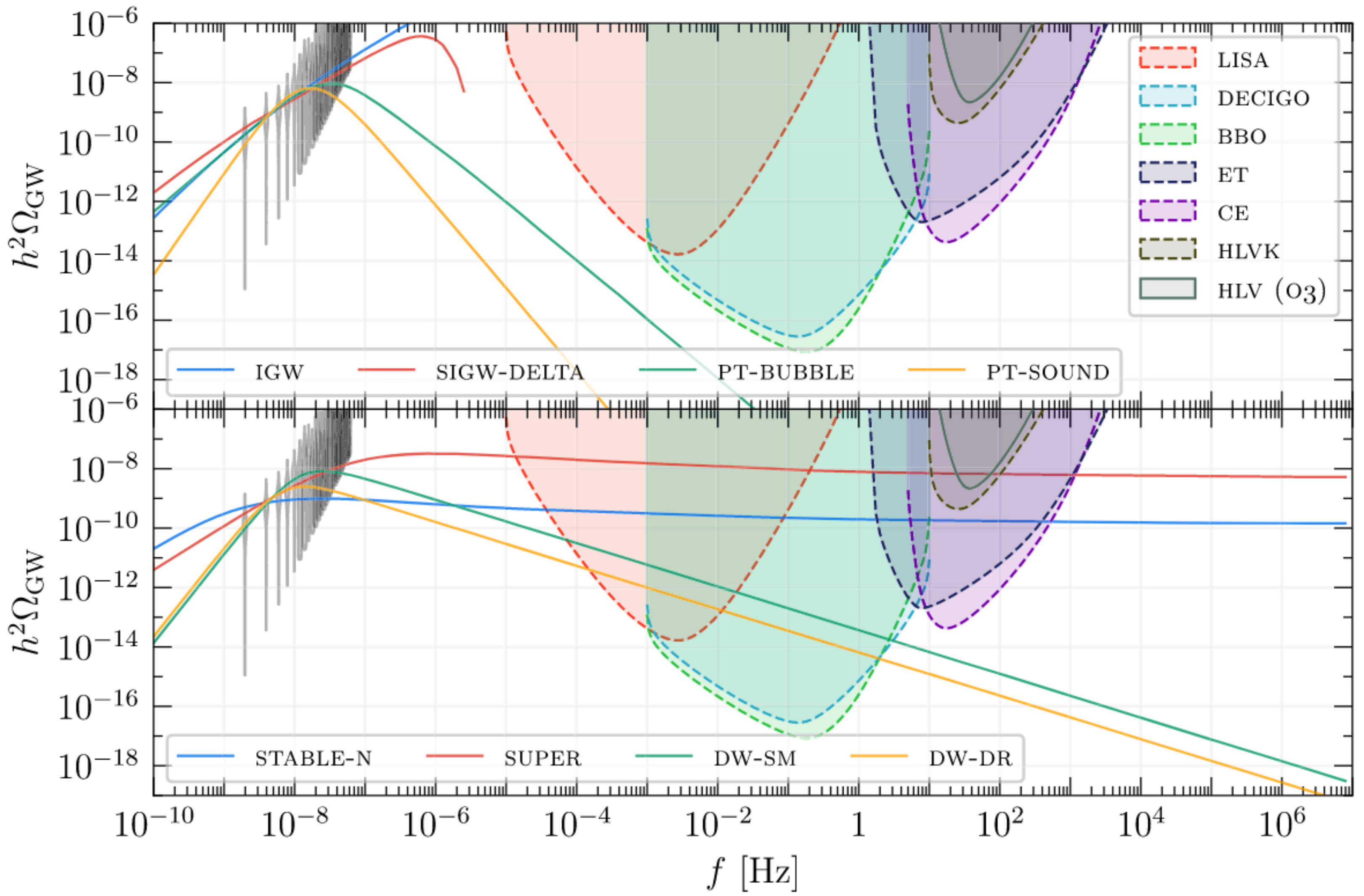
- **Non-Abelian BSM gauge sector** with deconfining high-T transition
- 1st order transition leads to **Gravitational Wave background**
- **Could be detectable** at GW telescopes

Symplectic Gauge Theories

- Recent interest in BSM models (with fermions)
- Give a minimal realisation of some **Dark Matter models**
- models of **composite Higgs with partially composite top**

Predictions require non-perturbative input $\Rightarrow$ Lattice methods!

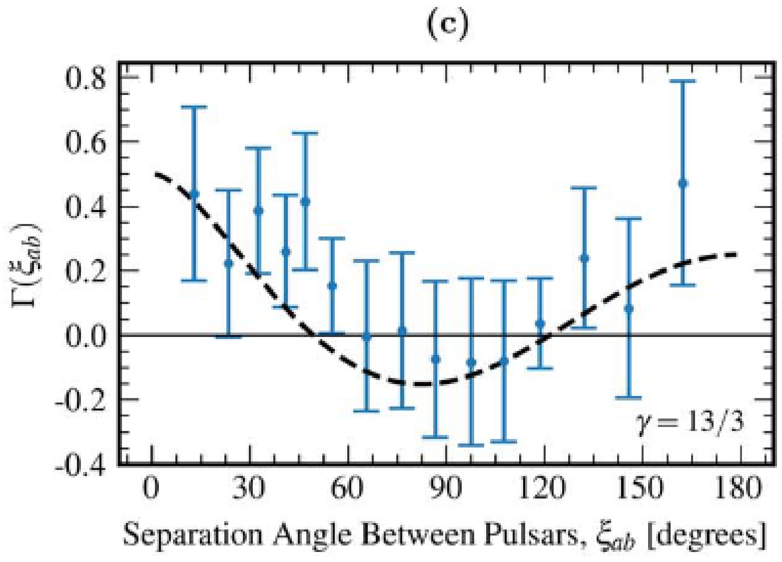
Prospects for experimental input



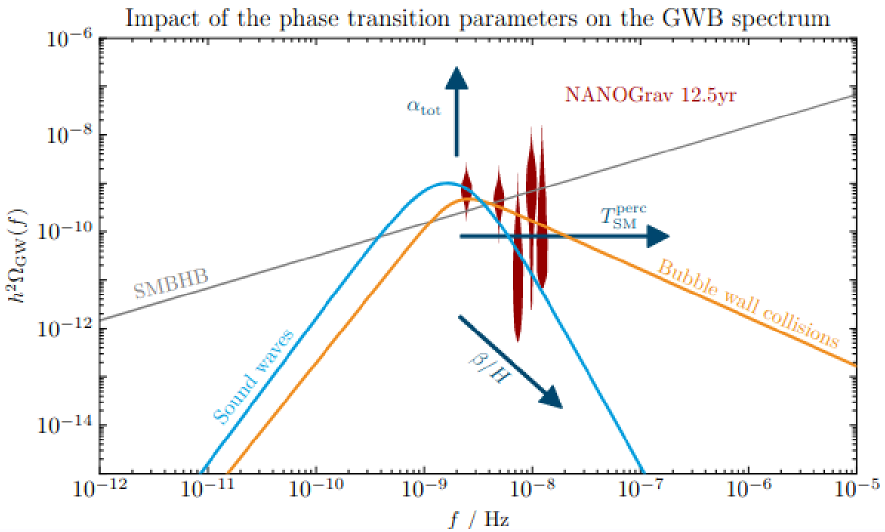
Afzal et al., [2306.16222] (astro-ph.HE)

Data is being collected right now

- NANOGrav released first analysis of dataset of 15yr observation period
- Consistent with a GW background (reported at ~$2-3\sigma$)
- Too early to establish existence or cause



Afzal et al., [2306.16213]



Bringmann et al., [2306.09411]

Sp(2N), Sp(6) see e.g. Wiese et.al. [hep-lat/0312022] G2: Pepe, Wiese [hep-lat/0610076] Cossu et.al. [0709.0669]

SU(N) see e.g. Lucini et.al. [hep-lat/0206029] [1202.6684] Panero [0907.3719] Rindlisbacher et.al. [2502.01396]

First-order phase transitions in gauge theories

- Finite temperature confinement-deconfinement transitions
- Order parameter typically: Polyakov loop (at least approximate)
- **Typically first order** except for $SU(2)$
- $Sp(2N > 4)$, $SU(N > 2)$, G_2 are first order ($SO(N)$ unknown)
  - Larger $N_c \to$ stronger phase transition
- With sufficiently heavy fermions: Still a first order transition

Our approach is applicable to generic gauge groups!

Where to expect 1st order transitions?

- Pure gauge with $N_c > 2$
- Deconfinement transitions for heavy fermions: $SU(3)$ with $N_f = 2$ has $m_\pi / T_c \approx 18(4)$ (corresponds to $m_\pi \approx 4$ GeV)
- Probably **no 1st order *chiral*** transition for all $N_f < N_f^{cw}$ below conformal window
- **Here: pure Yang-Mills** theory

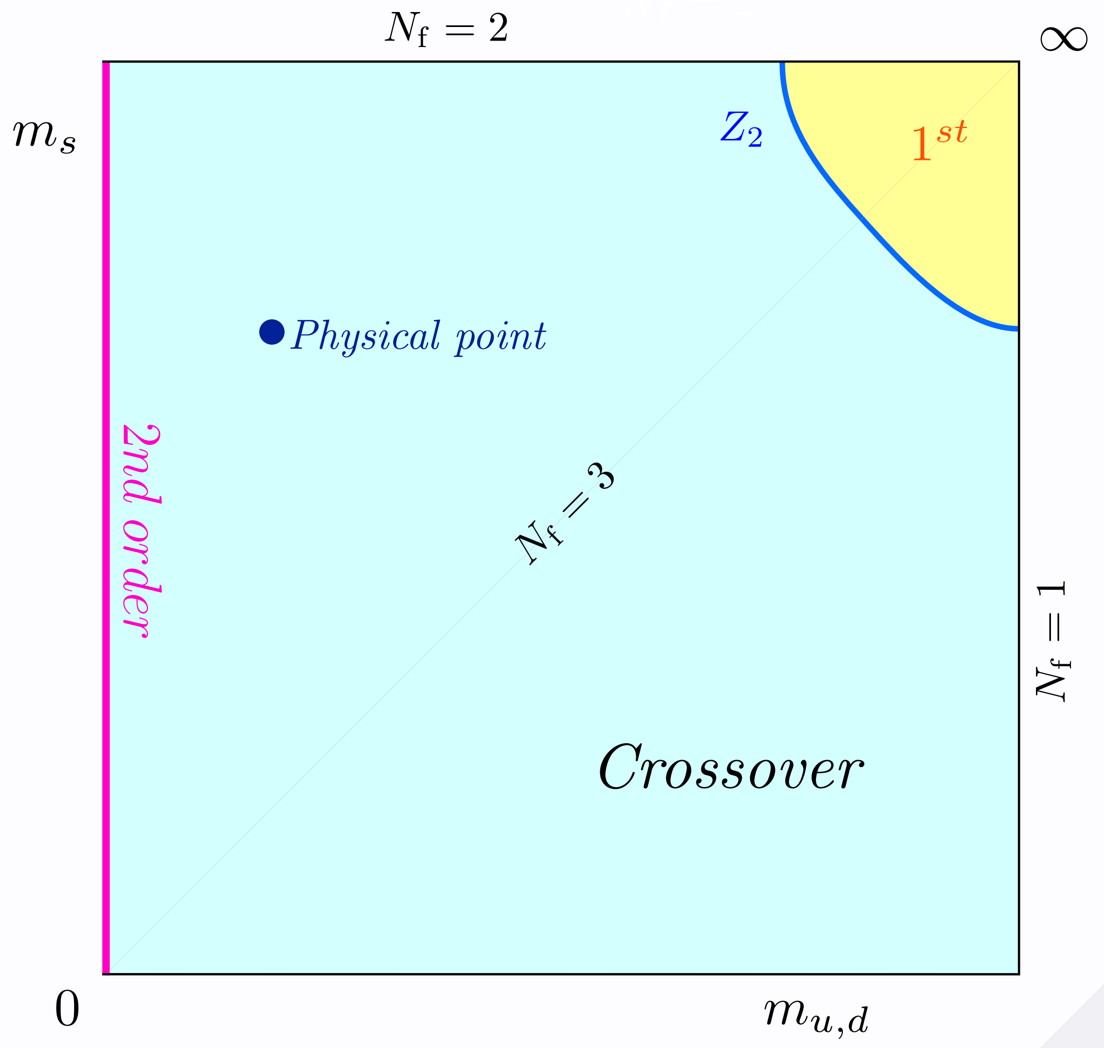


image: Cuteri et.al. [2107.12739][6]

Finite temperatures on the lattice

$$Z = \int D[A] \exp(-\beta S[A])$$
$$= \int D[A] \exp(-F_{\mu\nu}[A] F_{\mu\nu}[A] / g^2)$$

temperature $1/T = a(\beta) N_t$

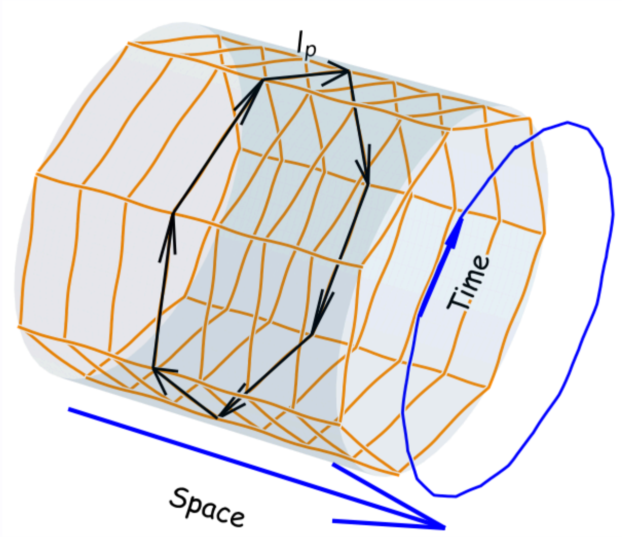


image courtesy of D. Mason

- Wilson action: link variable U, lattice spacing a, $V/a^4 = N_t N_s^3$
  $S[U] = 6V(1 - \underbrace{u_p[U]}_{\text{average plaquette}})$
- Relvant Observables: surface tension σ_{cd}, latent heat L_h

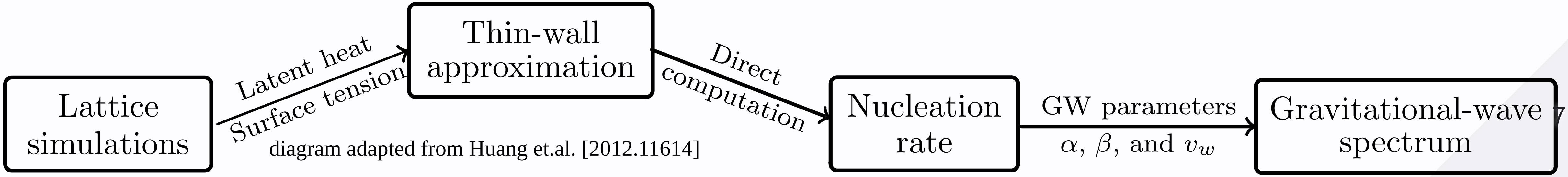


diagram adapted from Huang et.al. [2012.11614]

1st order transitions & importance sampling

- probability $P(E)$ has two peaks (phase coexistence)
- little tunneling between phases
- worsens as $V \to \infty$ and a
- Relates to surface tension σ_{cd} and latent heat L_h

$$\frac{P_{\min}}{P_{\max}} \propto \sqrt{N_s} \exp\left(-2\frac{N_s^2}{N_t^2}\frac{\sigma_{cd}}{T_c^3}\right)$$

$$\Delta\langle u_p\rangle_{\beta_c} \propto L_h/T_c^4$$

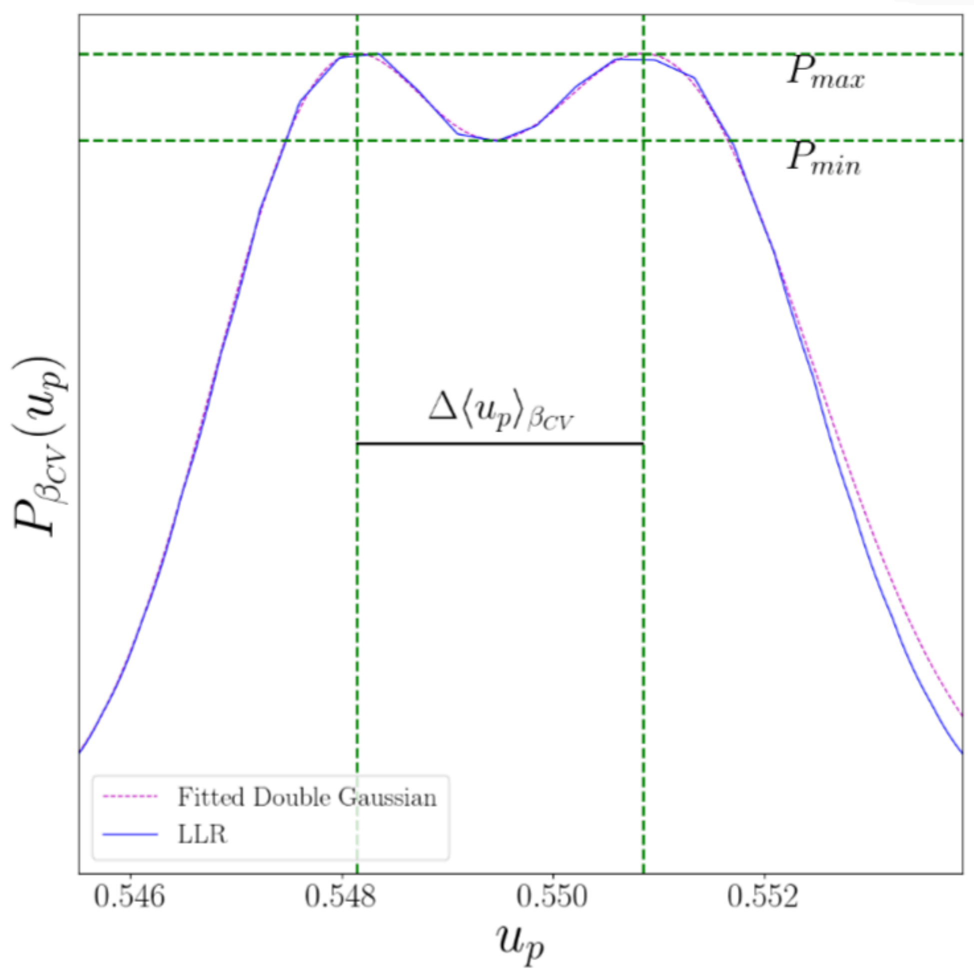


image: thesis D. Mason

Density-of-states approaches

- Microcanoical approach

$$Z = \int D[A] e^{\beta S[A]} = \int \mathrm{d}E \rho(E) e^{\beta S[A]}$$

$$\rho(E) = \int D[A] \delta(S[A] - E)$$

- $\rho(E) \Leftrightarrow$ any observable $O(E)$
- Determine $\rho(E)$ for fixed E: Avoid importance sampling problems
- Operators that are functions of the energy $E = S[A]$

$$\langle O(E) \rangle = \frac{1}{Z} \int \mathrm{d}E\ \rho(E) O(E) e^{\beta E}$$

- can be generalized to arbitrary operators $O[A] \neq O(E)$
- lattice: link variables $A \to U$

see e.g. Langfeld et.al. [1204.3243] [1509.08391]

Determining the $\rho(E)$: Linear Logarithmic Relaxation (LLR)

$$\rho(E) = \rho(0) \exp\left[-\int_0^E \mathrm{d}\bar{E}\, a(\bar{E})\right] \approx \exp\left[c^{(n)} + a^{(n)}(E - E_n)\right]$$

- Determine $a(E)$ in discrete intervals $E \in E_n$ of width δ
- Restrict action $S[U]$ to energy interval for given $a(E_n)$

$$\langle\langle O \rangle\rangle(\hat{a}) = \int D[U] O[U] W(S[U] - E_n, \delta) \exp\left[(S[U] - E_n)\,\hat{a}\right]$$

$$W(x, \delta) = \begin{cases} 1 & \text{if } x \in [-\delta/2, +\delta/2] \\ 0 & \text{else} \end{cases}$$

- The correct logarithmic density-of-states $a(E)$ fulfils

$\boxed{\langle\langle S[U] - E_n \rangle\rangle(\hat{a} = a(E)) = 0}$ $\Rightarrow$ restricted E expectation values see e.g. Langfeld et.al. [1204.3243] [1509.08391]

Robbins-Monro (RM) iterative method

$$\langle\langle S[U] - E_n \rangle\rangle(a) = 0$$

- *a stochastic Newton method*

$$\boxed{a_{n+1} = a_n - c_n \langle\langle S[U] - E_n \rangle\rangle(a_n)}$$

- condition on coefficient

$$\sum_n^\infty c_n = \infty, \qquad \sum_n^\infty c_n^2 \to \text{finite}$$

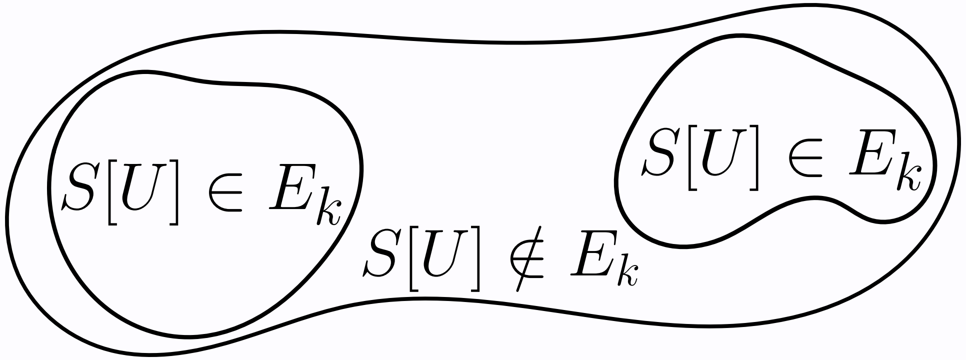
1. Start with initial a_0
2. Update with RM n times $\Rightarrow a_n$
3. Repeat N times $\Rightarrow \left\{a_n^{(i)}\right\}$
4. Measure observables: error estimate from N repetitions

Lattice setup: HiRep extended to $Sp(2N)$, heatbath updates with domain decomposition, studies of $SU(N)$ group are work-in-progress

Preserving ergodicity and approaching the continuum

- $\langle\langle O \rangle\rangle$ in restricted interval
- applies to entire Markov chain

cannot $\Rightarrow$ ergodicity issues



- Multiple intervals *in parallel*
  - allow replica to leave interval by swapping with neighbouring intervals

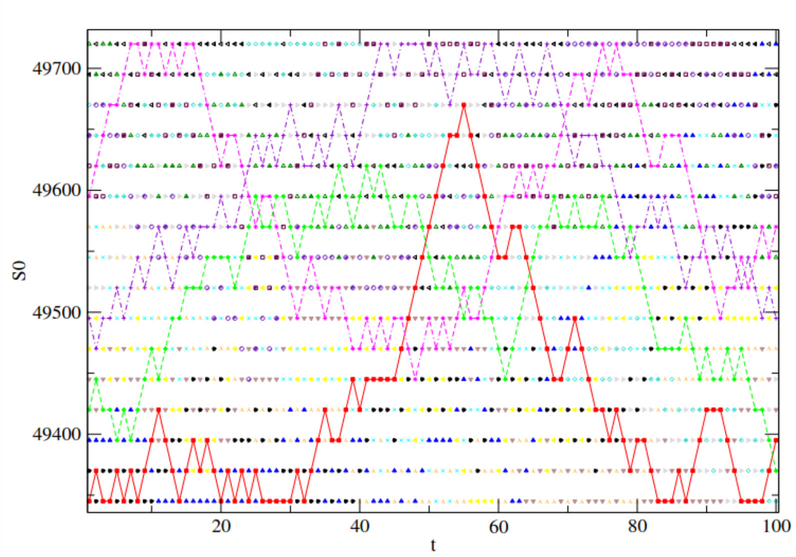


image: Pellegrini et.al. (LATTICE2016)

Required limits:

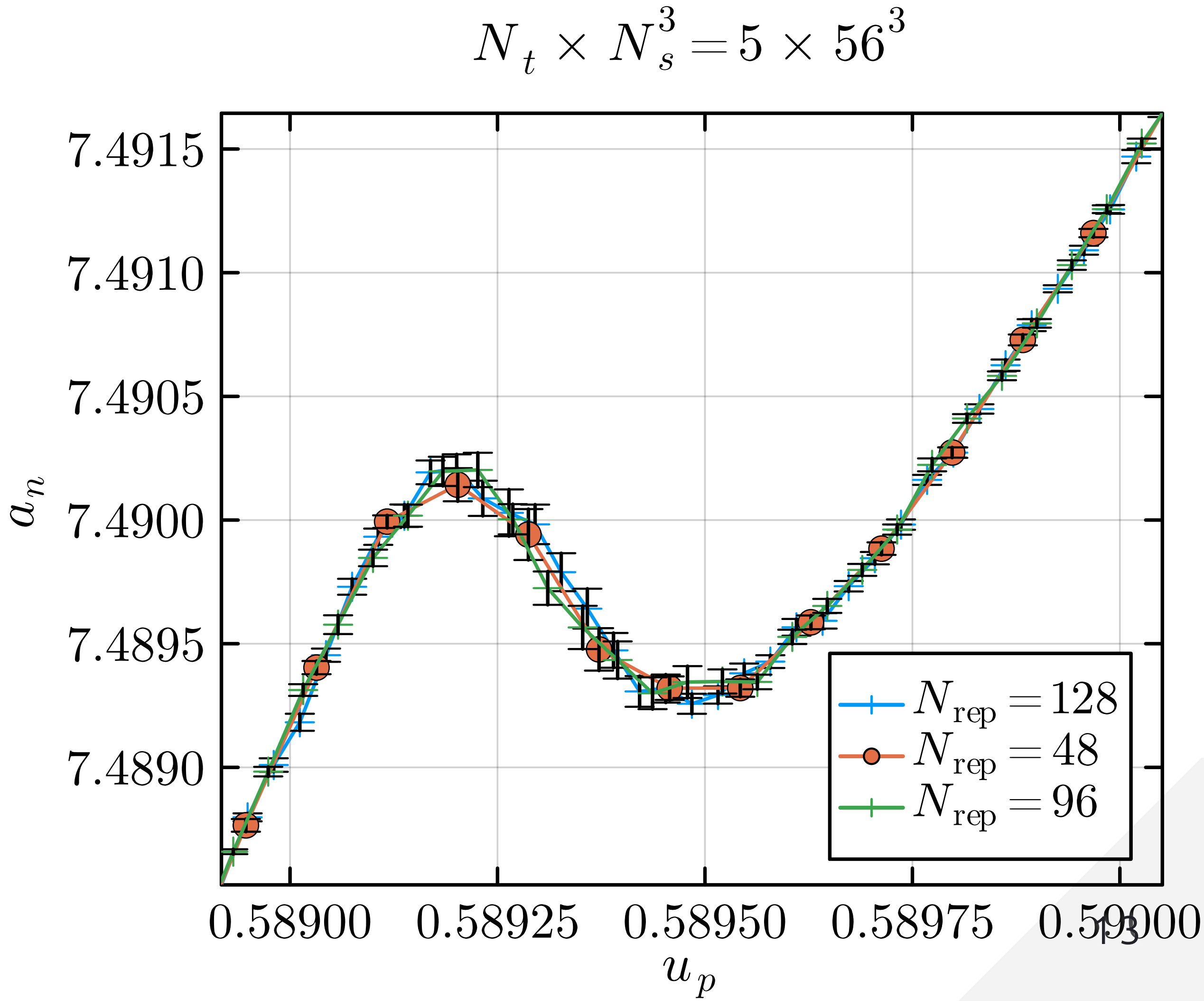
1. energy interval width $\delta \to 0$
2. thermodynamic limit $N_s \to \infty$
3. continuum $a \to 0 \Leftrightarrow N_t \to \infty$

Results: $Sp(4)$ **with** $N_t = 4, 5$

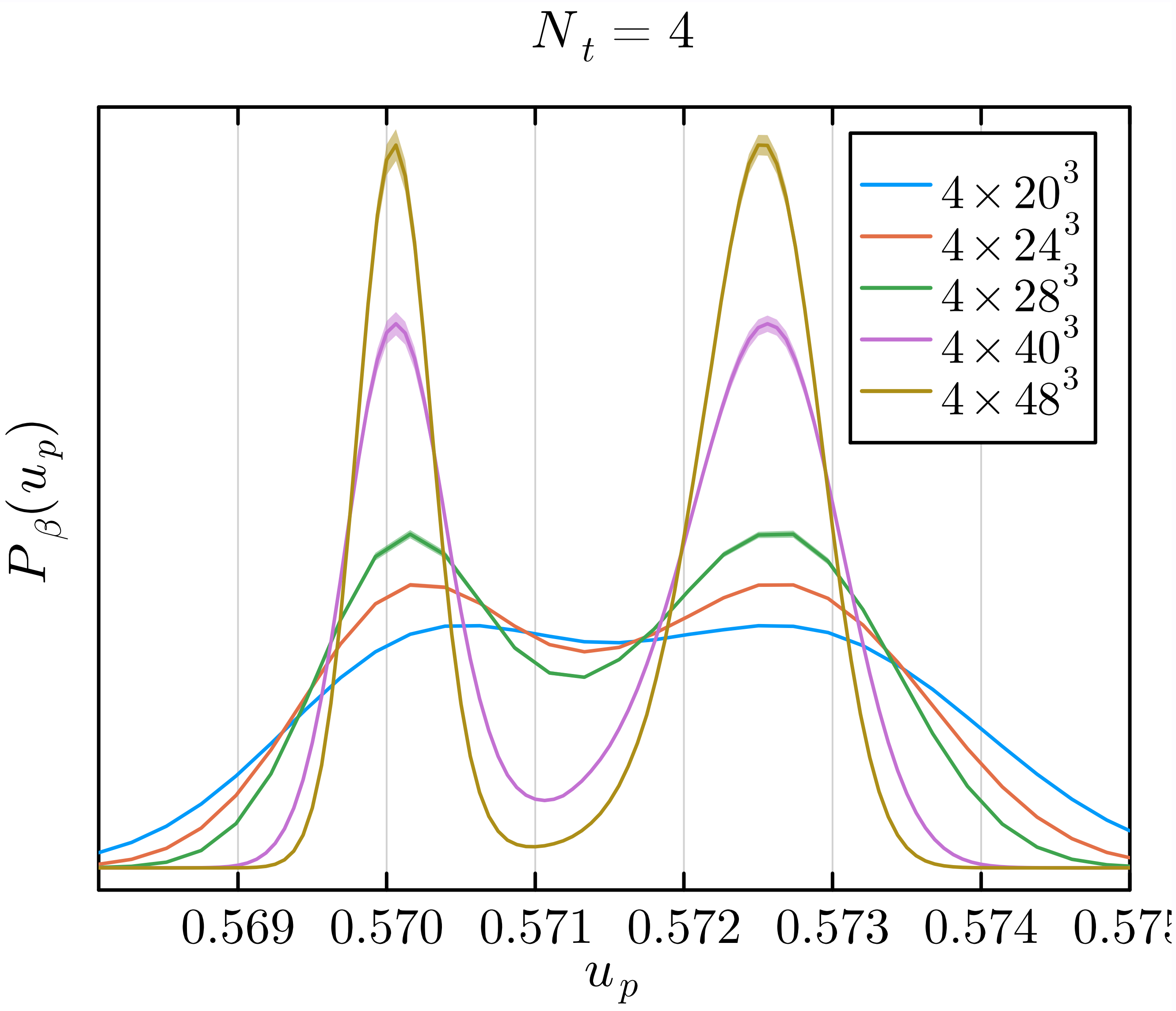
parameters studied

N_t	N_s	$u_p^{\min}$	$u_p^{\max}$	N_{rep}	N_{repeats}	n_{NR}	n_{RM}
5	48	0.588	0.592	48	25	10	60
5	48	0.588	0.592	96	25	10	50
5	56	0.588	0.592	128	25	10	50
5	56	0.588	0.592	48	25	10	50
5	56	0.588	0.592	96	25	10	50
5	64	0.588	0.592	95	20	7	50
5	72	0.588	0.592	95	20	11	50
5	80	0.588	0.59	64	20	15	30
4	20	0.565	0.58	64	20	10	300
4	24	0.565	0.58	64	20	10	300
4	28	0.565	0.58	64	20	10	200
4	40	0.568	0.576	128	25	10	100
4	48	0.568	0.576	128	26	10	100

varying energy interval width δ



$Sp(4)$ with $N_t = 4$ up to $N_s = 48$



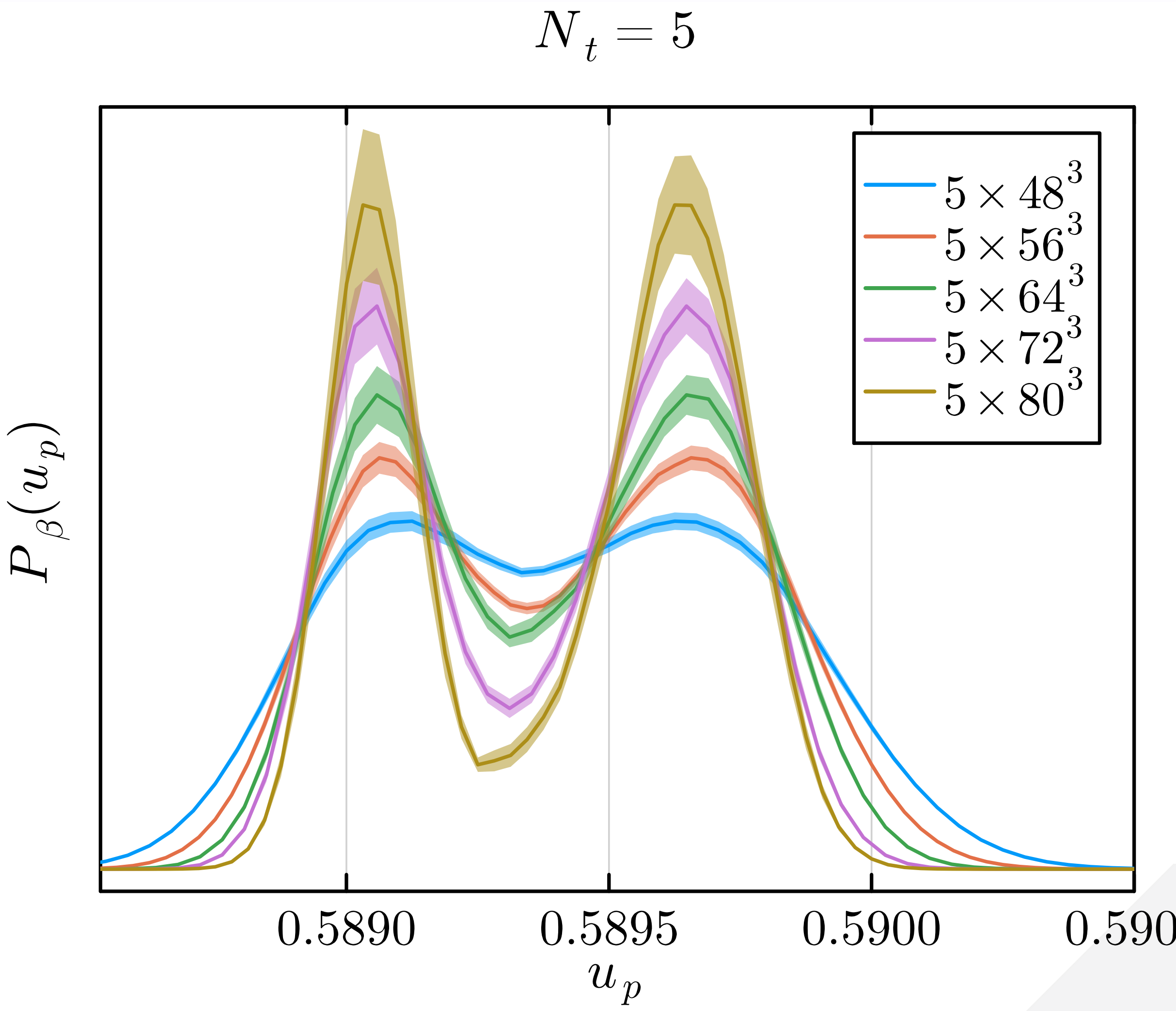
- clear double-peak structure ⇒ 1st order!
- β_c determined by requiring equal heights of peaks
- ratio $N_s/N_t > 5$ required
- no clear sign of plateau (=interface) at $N_s/N_t = 12$

$$\beta_c = 7.340084(6)$$

$Sp(4)$ with $N_t = 5$ up to $N_s = 80$

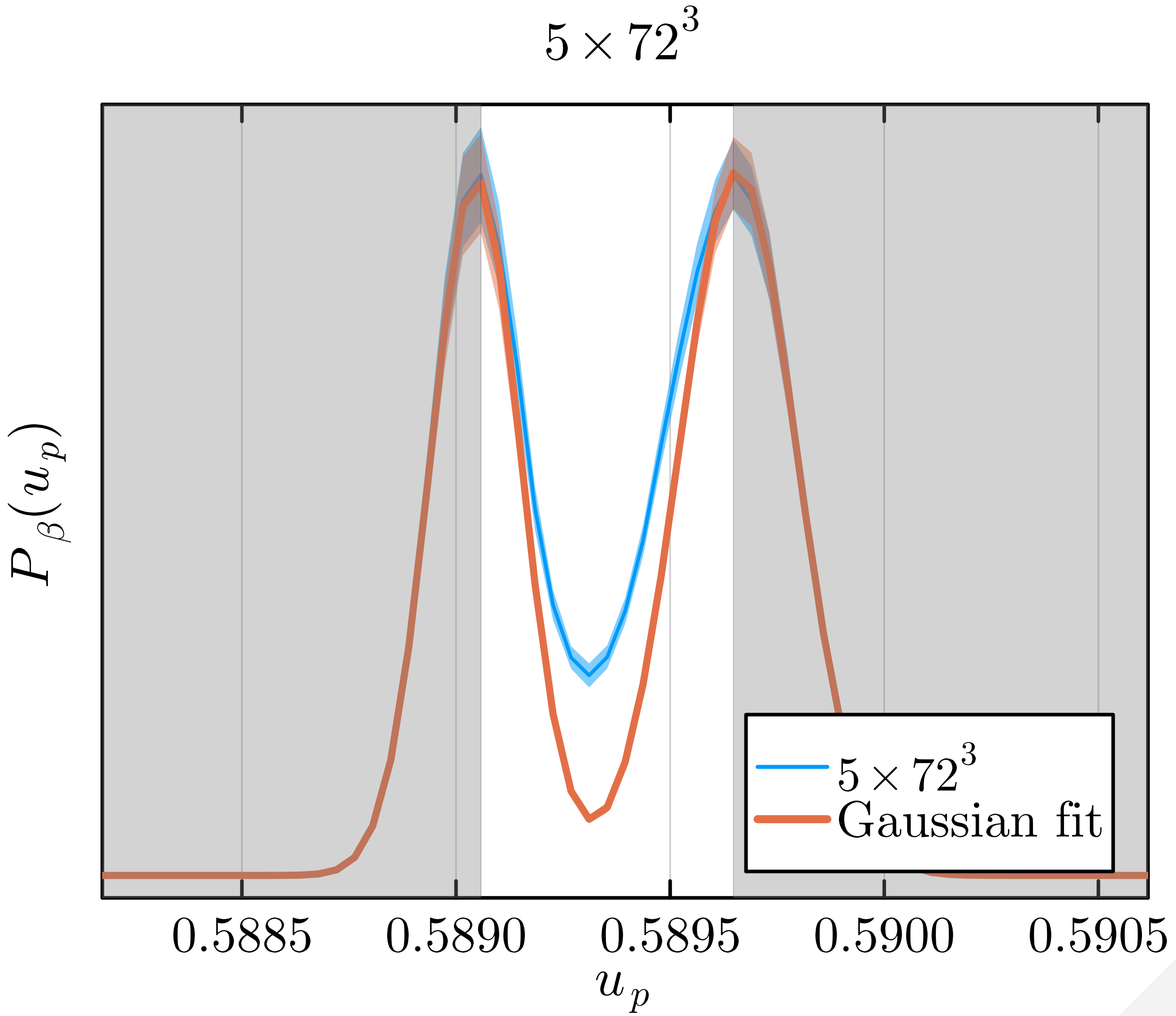
- same structure observed
- larger aspect ratios required $(N_s/N_t)^{\mathrm{min}} = 9.6$ to see the first order behaviour
- generally larger uncertainties, but still $\beta_c = 7.48969(4)$

  trend persists at $N_t = 6$
- we cannot resolve a double peak structure on 6×72^3

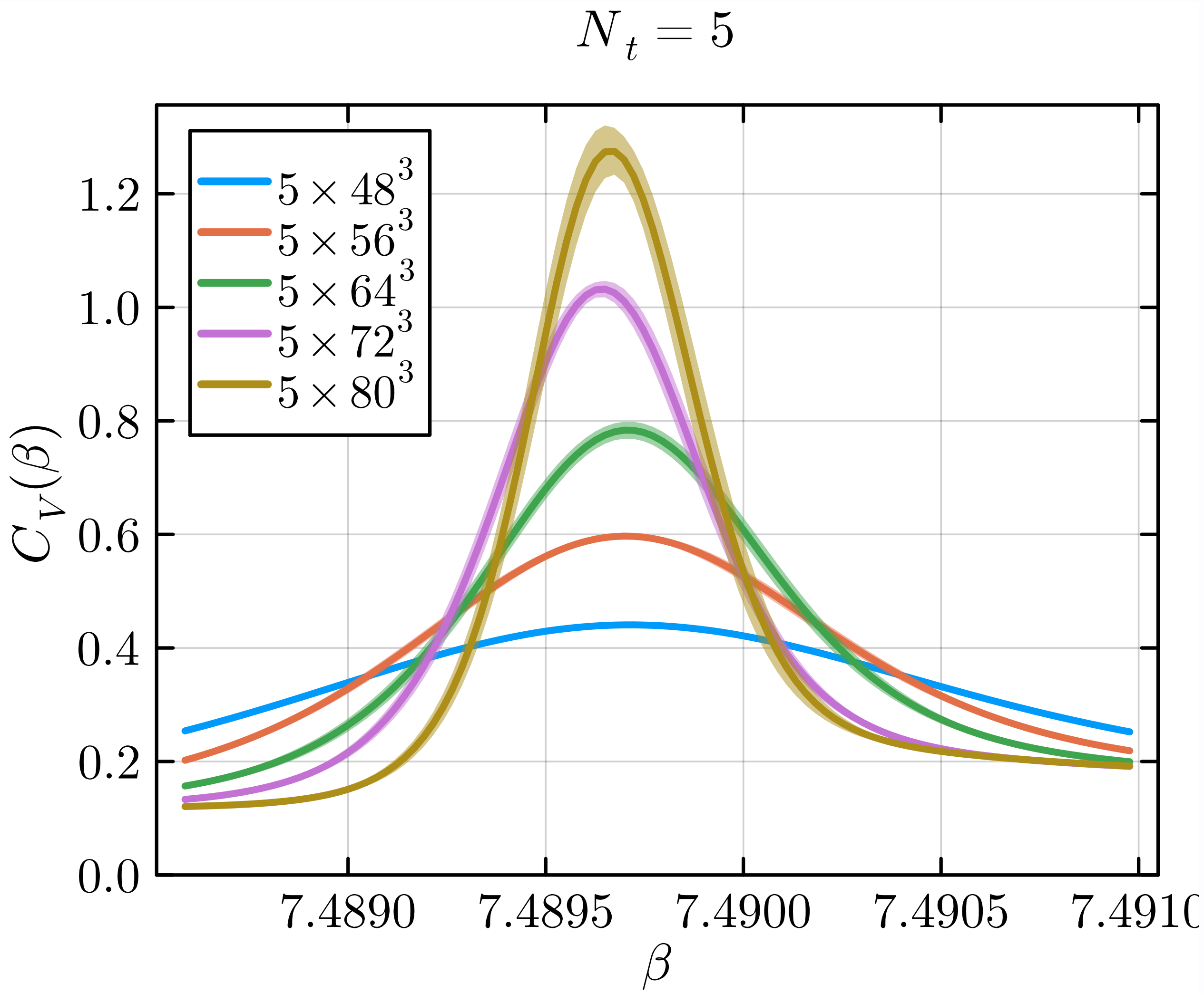


But the effects of an interface are present

- fit data two a bimodal Gaussian distribution
- difference in coexistence regime due to interface
- can be used to estimate surface tension



Critical coupling β_c from specific heat



- Alternative determination of critical coupling

$$C_V(\beta) \equiv \frac{6V}{a^4}\left(\langle u_p^2\rangle_\beta - \langle u_p\rangle_\beta^2\right)$$

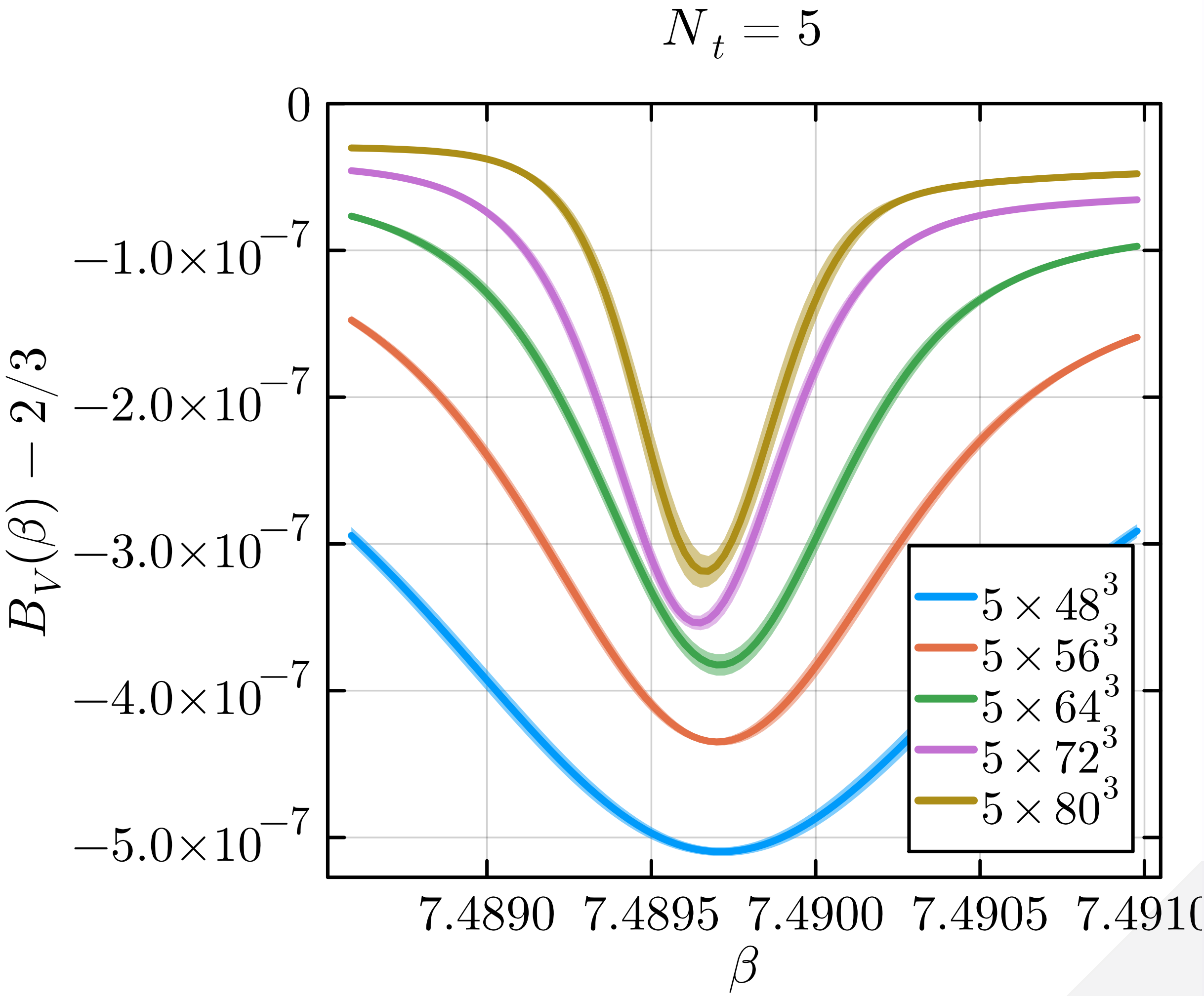
- has maximum at $\beta = \beta_c$
- no tuning of plaquette histogram required

Critical coupling β_c from Binder cumulant

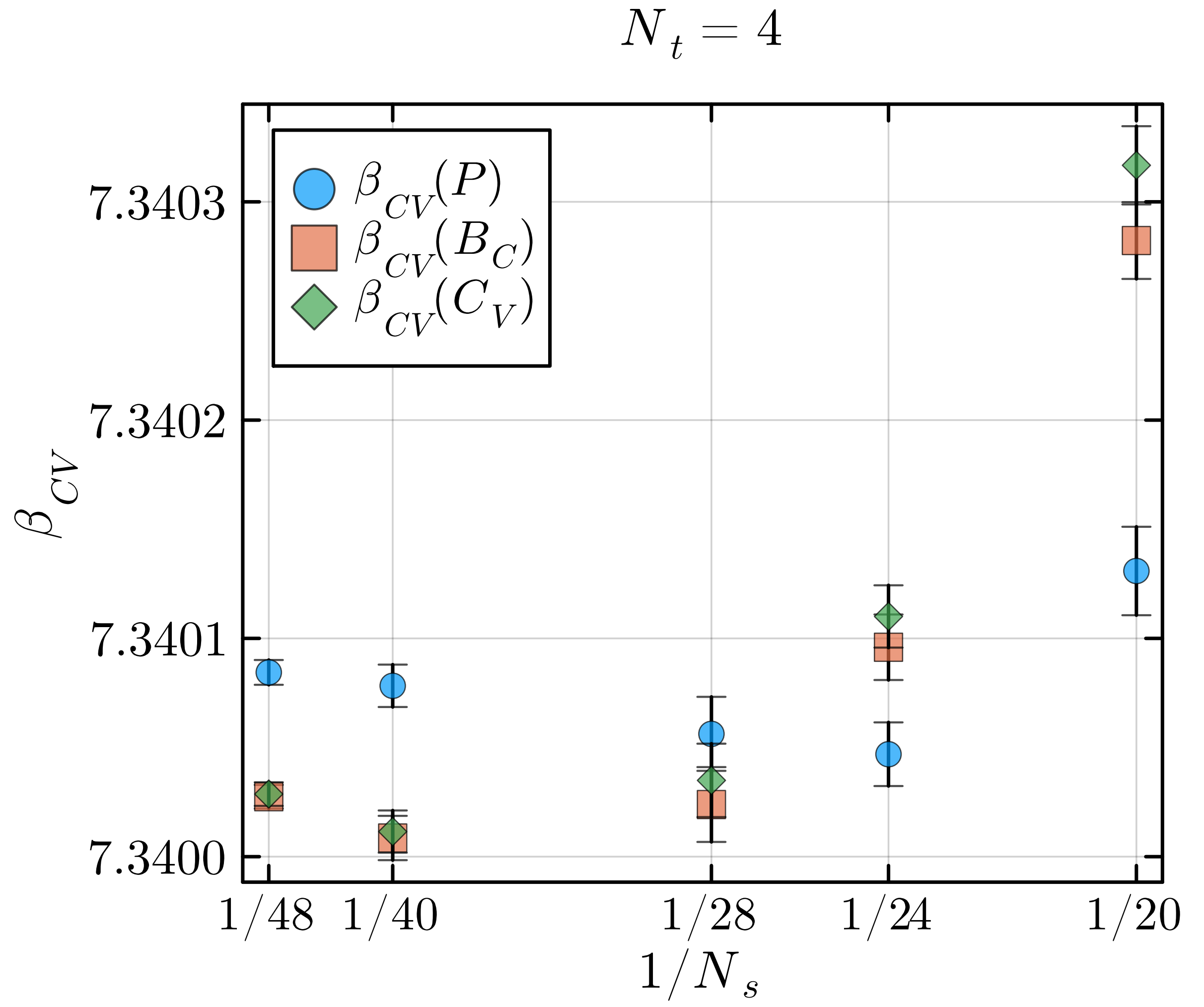
- Similar approach to C_V with higher order moments

$$B_V(\beta) \equiv 1 - \frac{\langle u_p^4 \rangle_\beta}{3\langle u_p^2 \rangle_\beta^2}$$

- minimum at criticality
- off criticality B_V tends to $2/3$
- numerical results agree well with C_V



Comparison: Determining critical β_c at $N_t = 4$

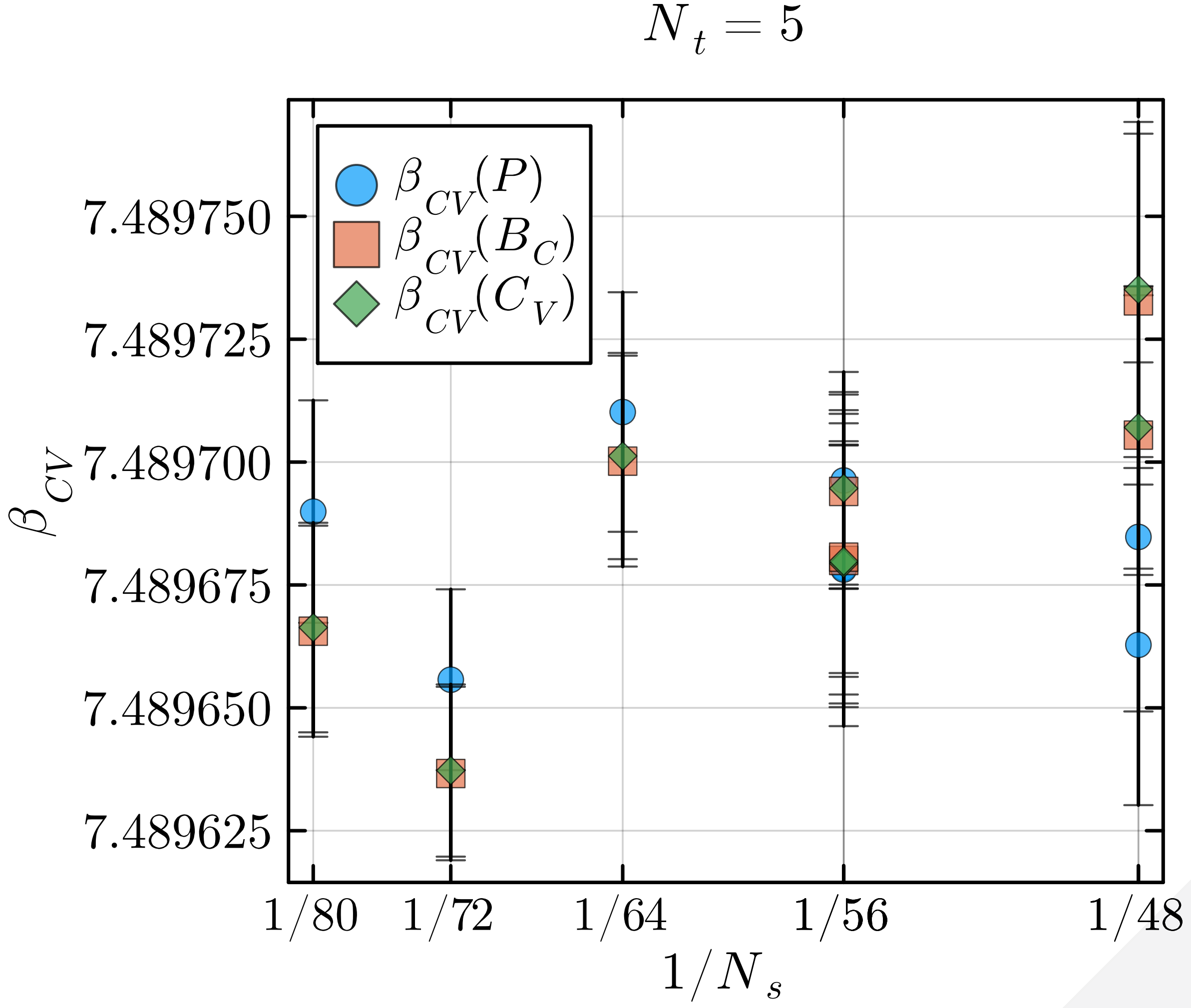


- strong volume dependence observed
- systematic differences between distribution and cumulant methods
- effects probably due to finite lattice spacing and overall high precision

Comparison: Determining critical β_c at $N_t = 5$

- much more consistent picture
- all volumes are rather large, and all methods agree well
- no substantial volume dependence is observed

$$\frac{\beta_c^{N_t=4}}{\beta_c^{N_t=5}} \approx 1.02$$

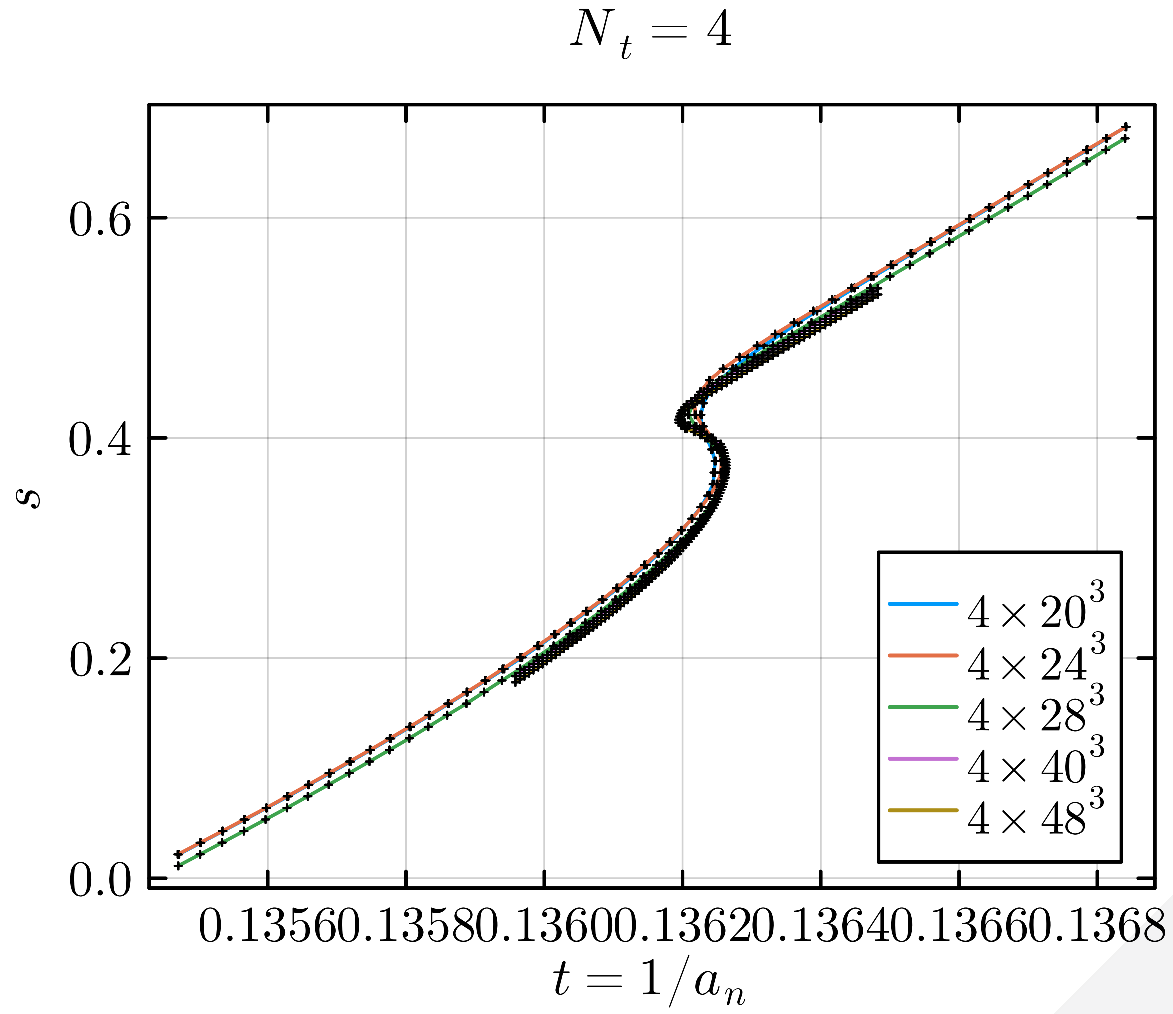


Other observables: entropy

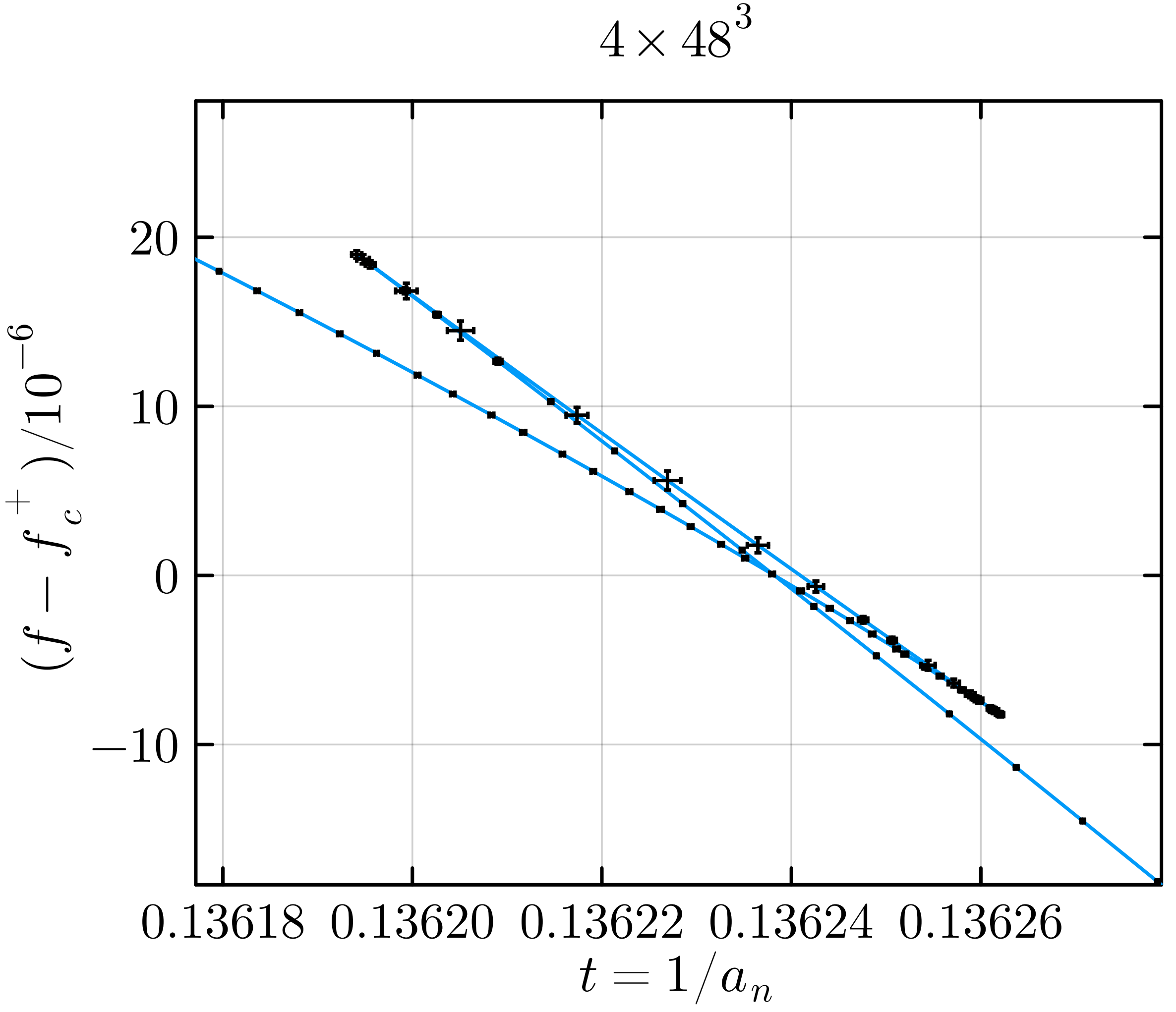
- Density of states related to entropy $s = \log(\rho)$
- We know ρ up to a constant

$$\log(\rho)(E) \approx c^{(n)} + a^{(n)}(E - E_n)$$

- Fix s at β_c, such that $s > 0$
- This violates $\lim_{T\to 0} s = 0$ (requires $T \to 0$ limit of ρ)



Other observables: free energy

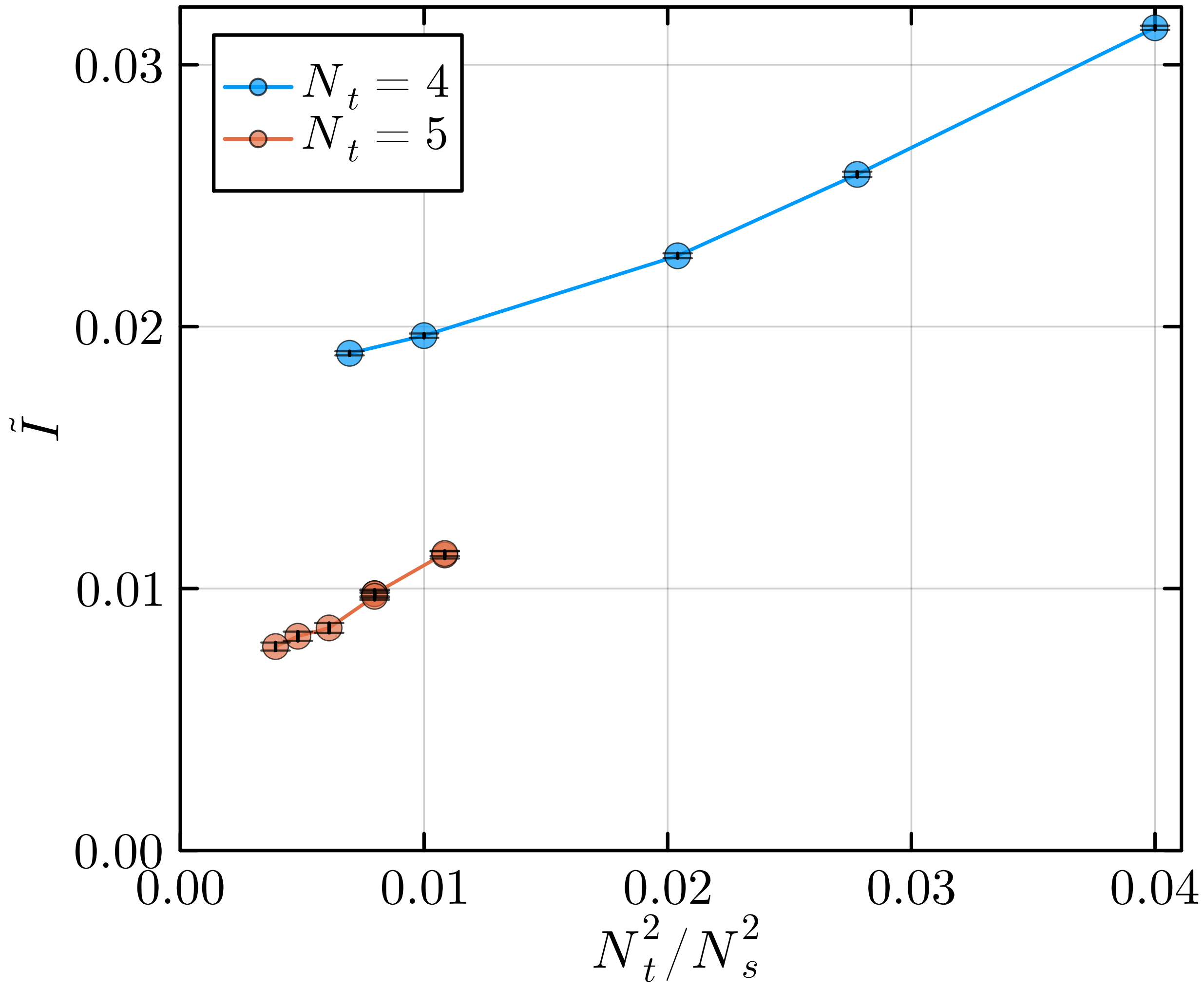


- Free energy f from Legendre transform

$$f = ts - \frac{E}{V}$$

- $t = 1/a_n$ micronanoical temperature
- phase coexistence from *swallow-tail* form of f
- unknown constant in $f \Leftrightarrow$ to unknown slope of f

Surface tension



- Determined from

$$\tilde{I} = -\frac{1}{2}\left(\frac{N_t}{N_s}\right)^2 \log\left(\frac{P_{\min}}{P_{\max}}\right) + \frac{1}{4}\left(\frac{N_t}{N_s}\right)^2 \log(N_s)$$

- approaches dimensionless surface tension for large ratios

$$\lim_{N_s/N_t \to \infty} \tilde{I} = \frac{\sigma_{cd}}{T_C^3}$$

- strong discretization artefacts
- current results probably upper limit on $\tilde{I}$ in continuum

Summary

- First order transitions occur for heavy fermions and in pure gauge
- Standard Lattice techniques struggle with 1st order transitions
- LLR provides an alternative approach to perform first-principles lattice calculations!
- More work needed to take the full continuum limit

Thank you